Symmetrised Kronecker products of the fundamental representation of Sp(n, R)

1990 J. Phys. A: Math. Gen. 23 1909

(http://iopscience.iop.org/0305-4470/23/11/017)

J. Phys. A: Math. Gen. 23 (1990) 1909-1927. Printed in the UK

Symmetrised Kronecker products of the fundamental representation of Sp(n, R)

M J Carvalho

Department of Mathematics, Physics and Computer Science, Ryerson Polytechnical Institute, Toronto, Ontario, Canada M5B2K3

Received 14 November 1989

Abstract. The role played by S-function series in the character theory of the non-compact symplectic group Sp(n, R) is exploited to derive general formulae for the evaluation of symmetrised Kronecker products of the fundamental Sp(n, R) representation. Applications of the results to Sp(3, R) and their impact on the nuclear symplectic shell model (SSM) are discussed for simple cases.

1. Introduction

The importance of group characters in the theory of group representations is well established. In particular, the characters of the unitary groups, for which analytic expressions are given by Weyl's celebrated character formula (Weyl 1939), play a central role in the theory of both Lie group and the symmetric group. U(n) characters are homogeneous symmetric polynomials in the characteristic values $(\alpha_1, \ldots, \alpha_n)$ of the U(n) matrices. They are known from the theory of symmetric functions as Schur functions or S-functions (Littlewood 1940).

Much is known about the properties of S-functions. For example, infinite S-function series, first introduced by Littlewood (Littlewood 1940), have since been extensively studied (King 1975, King *et al* 1981, King and Wybourne 1982, Black *et al* 1983) and shown to play a fundamental role in the evaluation of branching rules and Kronecker products for compact Lie groups.

For non-compact Lie groups, Rowe *et al* (1985) have shown that the characters of the positive discrete series representations of metaplectic Sp(n, R) are expandable as infinite series of U(n) characters, or equivalently, S-functions. Thus, the branching rule

$$\mathrm{Sp}(n, R)\downarrow \mathrm{U}(n)$$

is expressible in terms of already familiar infinite S-function series.

A systematic presentation of S-functions series was given by Yang and Wybourne (1986) who also suggested that symmetrised Kronecker products of Sp(n, R) irreps may require the construction of new series of S-functions.

This paper attempts to evaluate the irreducible symplectic content of the symmetrised m-power of the fundamental Sp(n, R) irrep. In the process, the S-function content of new generating polynomial functions is identified.

The motivation for this problem stems from earlier work done on the nuclear 'symplectic shell model' (SSM) (Rowe 1985, Carvalho *et al* 1986) and is of importance for its implementation.

The symplectic shell model derives its success from the fact that it is the appropriate model for a many-body description of collective motion either in phenomenological or microscopic terms. In this model, the configuration space for a given A-particle nucleus is decomposed into collective subspaces which carry irreps of Sp(3, R). Phenomenological calculations in the SSM only require a knowledge of the basis of the Sp(3, R) irreps they belong to. On the other hand, microscopic calculations require further identification of the basis states in harmonic oscillator shell model terms.

Up to now the model has been mostly applied to nuclei whose collective properties are reasonably well described by restricting the full space to only one collective subspace. However, it has been suggested (Rowe 1985, Carvalho *et al* 1986) that in order to explain the shell model structure of the so-called beta and gamma vibrational bands in deformed nuclei, one should take into account more than one collective space. Full application of the model then requires a knowledge of the collective spaces (i.e. Sp(3, R) irreps) occurring in the shell model of a given nucleus and their relative importance.

In order to obtain the Sp(3, R) irreps which correspond to the collective subspaces of an A-particle nucleus, one has to determine the Sp(3, R) irreps occurring in the decomposition of the product of A copies of the fundamental representation of Sp(3, R).

Because Sp(3, R) operators are fully symmetric one-body operators, they preserve particle symmetry. Therefore each collective subspace has associated with it a definite permutation symmetry. So, if one is only interested in determining the collective spaces, of the A-particle nucleus, which are of a given symmetry type, it suffices to evaluate the corresponding symmetrised A-power of the fundamental Sp(3, R) irrep.

The plan of the paper is as follows. Section 2 summarises those aspects (definitions and properties) of the S-function formalism which are relevant for the development of following sections. Section 3 reviews the role played by S-functions in the character theory of the non-compact group Sp(n, R). In section 4 one derives general formulae for the evaluation of symmetrised products of the fundamental irrep of Sp(n, R), and in section 5 the results obtained in section 4 are particularised to systems of two and four particles in Sp(3, R).

Note that the work presented in this paper, though of special relevance to the nuclear symplectic shell model, is of much more general importance. A method is given for the explicit evaluation of symmetrised Kronecker products of the fundamental representation of non-compact symplectic groups, of arbitrary dimension, and the formulae obtained are in suitable form for automatic computation.

2. S-functions and U(n) irreps

A standard S-function, labelled by $\{\lambda\}=\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ (with (λ) a partition in n parts, i.e. a set of non-negative integers $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \ldots \geqslant \lambda_n$) is a symmetric function of a set of indeterminates $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ given by

$$\{\lambda\} = \det(\alpha_t^{\lambda_s+n-s})/\det(\alpha_t^{n-s}) \tag{2.1}$$

where t and s index rows and columns, respectively, of the $n \times n$ determinants.

Alternatively, the determinants can be written as

$$\{\lambda\} = \frac{\sum \varepsilon(\pi)\alpha_{\pi_1}^{\lambda_1+n-1}\alpha_{\pi_2}^{\lambda_2+n-2}\ldots\alpha_{\pi_n}^{\lambda_n}}{\sum \varepsilon(\pi)\alpha_{\pi_1}^{n-1}\alpha_{\pi_2}^{n-2}\ldots\alpha_{\pi_n}^{0}} \tag{2.2}$$

where $\varepsilon(\pi)$ is the parity of the permutation $\pi \equiv \pi_1\pi_2\ldots\pi_n$ and the sum is over all elements of the symmetric group S_n.

It can be easily checked that the denominator in (2.2) is a factor of the numerator, and therefore the S-function is a sum of monomials in the α_i all of degree $|\lambda| = \lambda_1+\lambda_2+\lambda_3+\ldots+\lambda_n$, referred to as the weight of the S-function.

Non-standard S-functions, for which the condition $\lambda_1 \geqslant \ldots \geqslant \lambda_n$ is not satisfied, are either zero or convertible to standard ones. The well known modification rule (Murnaghan 1938, Littlewood 1940)

$$\{\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_i, \ldots, \lambda_n\} = -\{\lambda_1, \lambda_2, \ldots, \lambda_i - 1, \lambda_{i-1}+1, \ldots, \lambda_n\} \tag{2.3}$$

used to standardise an S-function whose entry $\lambda_{i-1} < \lambda_i$, follows directly from the determinantal definition (2.1). In fact, the rule (2.3) is a consequence of the property of determinants that interchanging two columns in a determinant merely changes the sign of its value. For example,

$$\{04\} = \frac{\begin{vmatrix} \alpha_1^1 & \alpha_1^4 \\ \alpha_2^1 & \alpha_2^4 \end{vmatrix}}{\begin{vmatrix} \alpha_1^1 & \alpha_1^0 \\ \alpha_2^1 & \alpha_2^0 \end{vmatrix}} = -\frac{\begin{vmatrix} \alpha_1^4 & \alpha_1^1 \\ \alpha_2^4 & \alpha_2^1 \end{vmatrix}}{\begin{vmatrix} \alpha_1^1 & \alpha_1^0 \\ \alpha_2^1 & \alpha_2^0 \end{vmatrix}} = -\{31\}.$$

By successive applications of the modification rule (2.3) any non-standard, non-zero, S-function can be converted into a standard one.

Non-standard S-functions which turn out to be zero are those for which the corresponding determinant has at least two identical columns. It is easy to see that the determinant in (2.1) has identical columns i and k when the ith entry, λ_i, of the S-function is related to the kth entry, λ_k, by $\lambda_i = \lambda_k + i - k$ for $k > i$.

As pointed out by Weyl (1939), S-functions establish the connection between the symmetric group and the unitary group.

In fact, the S-functions of weight $|\lambda| = N$ are in one-to-one correspondence with the irreducible representations of the symmetric group S_N. The correspondence is given by expressing the S-functions in terms of the characters of the symmetric group.

Defining the 'symmetric power sum' functions (Wybourne 1970), p_r, of the indeterminates α_i as

$$p_r = \sum \alpha_i^r \tag{2.4}$$

expansion (2.2) is straightforwardly expressed in the form

$$\{\lambda\} = (1/N!) \sum_\rho h_\rho \chi_\rho^{(\lambda)} S_\rho \tag{2.5}$$

where

(i)

$$S_\rho = p_1^{\nu_1} p_2^{\nu_2} \ldots p_n^{\nu_n} \tag{2.6}$$

for each class ρ,

(ii) the class

$$\rho = 1^{\nu_1} 2^{\nu_2} \ldots n^{\nu_n} \tag{2.7}$$

has ν_1 1-cycles, ν_2 2-cycles, ... ν_n n-cycles

(iii) h_ρ is the order of the class,

(iv) $\chi_\rho^{(\lambda)}$ are the characteristics of the particular irrep $\{\lambda\}$ of the symmetric group S_N.

On the other hand, the identification

$$\alpha_t = e^{\varepsilon_t} = e^{i\varepsilon_t} \tag{2.8}$$

turns the expansion (2.2) into Weyl's character formula for a covariant tensor representation (λ) of U(n) (the unitary group in n dimensions). So each irreducible representation (irrep) of U(n) is also uniquely identified by an S-function.

But S-functions are more important than merely for labelling representations. Their polynomial nature makes them amenable to operations such as addition, subtraction, multiplication (outer, inner and plethysm) and division. All these operations have their counterpart operations on representations of the group they characterise.

It is also their polynomial nature which underlies the symbolic rules established for the manipulation of S-functions and the group characters they represent. As an illustration of the above statement, note the following. The dimension of an irrep $\{\lambda\}$ of U(n) is n-dependent and given by

$$d_n^{\{\lambda\}} = \prod_{i,j} \frac{n-i+j}{\lambda_i + \tilde{\lambda}_j - i - j + 1} \tag{2.9}$$

where $(\tilde{\lambda}) = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n)$ is the partition conjugate to (λ) and $1 \leqslant i, j \leqslant n$ such that $\lambda_i - j \geqslant 0$ and $\tilde{\lambda}_j - i \geqslant 0$.

For example, from (2.1), the polynomial expansion of $\{21\}$ in U(2) and U(3) is given, respectively, by

U(2)

$$\{21\} = \alpha_1^2\alpha_2 + \alpha_1\alpha_2^2 = \sum_{i,j=1}^{2} \alpha_i^2\alpha_j \tag{2.10a}$$

U(3)

$$\begin{aligned}\{21\} &= \alpha_1^2\alpha_2 + \alpha_1^2\alpha_3 + \alpha_2^2\alpha_3 + \alpha_1\alpha_2^2 + \alpha_1\alpha_3^2 + \alpha_2\alpha_3^2 + \alpha_2\alpha_1\alpha_3 + \alpha_1\alpha_3\alpha_2 \\ &= \sum_{ij}^{3} \alpha_i^2\alpha_j + 2\alpha_1\alpha_2\alpha_3.\end{aligned} \tag{2.10b}$$

Since the dimension of a representation is given by evaluating its character on the identity element, one concludes that the dimension of a U(n) irrep is equal to the number of monomials terms in the corresponding S-function expansion. Thus the irrep $\{21\}$ has dimension 2 in U(2) and 8 in U(3) in accord with (2.9).

Groups other than the symmetric and the unitary groups have characters which are not, in general, single S-functions but rather finite or infinite expansions of S-functions. Convenient and compact expressions of such expansions make use of series of S-functions. For instance, a character $[\lambda]$ of the orthogonal group O$(n) \subset$ U(n) is given by

$$[\lambda] = \{\lambda\}/C \tag{2.11}$$

an algebraic sum of S-functions resulting from the division of $\{\lambda\}$ by each term of the series $C = \Sigma_\gamma (-1)^{|\gamma|/2}\{\gamma\}$ where $|\gamma|$ is the weight of $\{\gamma\}$ and the sum includes S-functions which in Frobenius notation are of the type

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ a_1 - 1 & a_2 - 1 & \ldots & a_r - 1 \end{pmatrix} \tag{2.12}$$

with $a_1 > a_2 > \ldots > a_r$, and r the Frobenius rank. The correspondence between the Frobenius notation

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix}$$

and the standard notation $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ is given by the relations $a_i = \lambda_i - i$ and $b_i = \tilde{\lambda}_i - i$ where $1 \leqslant r \leqslant k$ is such that a_r and b_r are non-negative. So, for example,

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

corresponds to $\{444\}$.

On the other hand, a character of a generic representation of $\mathrm{Sp}(n, R) \supset \mathrm{U}(n)$ is given by

$$\langle(\lambda)\rangle = \{\lambda\}.D \tag{2.13}$$

a sum of S-functions obtained in the (outer) product of $\{\lambda\}$ by each term of the series $D = \Sigma_\delta \{\delta\}$ where $\{\delta\}$ are partitions with only even parts.

The important series of S-functions that have been identified in such applications are known to have rather simple generating functions (Littlewood 1940, Yang and Wybourne 1986). For example, the generating functions of the series C and D are given, respectively, by

$$C = \prod_{i \leqslant j}^{n} (1 - \alpha_i \alpha_j) = \sum (-1)^{|\gamma|/2} \{\gamma\} \tag{2.14a}$$

$$D = \prod_{i \leqslant j}^{n} (1 - \alpha_i \alpha_j)^{-1} = \sum \{\delta\}. \tag{2.14b}$$

Another series which is of importance to this paper is the series M whose generating function is

$$M = \prod_{i}^{n} (1 - \alpha_i)^{-1} = \sum \{m\} \tag{2.14c}$$

where $\{m\}$ is any S-function in one part only.

As is clear from the above polynomial functions, the series C and D are inverses of each other, for a given n, i.e.

$$C.D = 1 = \sum (-1)^{|\gamma|/2} \{\gamma\}\{\delta\} = \{0\} = 1.$$

Note the two following features about series in general.

(i) For a finite value of n, where n is the number of indeterminates, a series is finite if and only if the indeterminates α_i do not appear in the denominator. So in (2.14) only C is finite for a finite value of n.

(ii) The maximum number of parts with which the S-functions can appear in a series is equal to n. An exception to this rule occurs for the series M and the members of the M family, namely

$$M^{\dagger} = \prod_{i}^{n} (1 + \alpha_i)^{-1} = \sum (-1)^m \{m\} \tag{2.15a}$$

$$M_+ = \tfrac{1}{2}(M + M^{\dagger}) = \sum_{m \text{ even}} \{m\} \tag{2.15b}$$

$$M_- = \tfrac{1}{2}(M - M^{\dagger}) = \sum_{m \text{ odd}} \{m\} \tag{2.15c}$$

which have only one part regardless of n.

3. Sp(n, R) and S-functions

Among the representations of the non-compact symplectic group Sp($2\sigma n$, R), the representation spanned by the states of the $2\sigma n$-dimensional harmonic oscillator plays a very important role in physical applications.

A realisation of the associated symplectic algebra in terms of creation and destruction harmonic oscillator operators is

$$A_{ij}^{st} = b_i^{\dagger s} b_j^{\dagger t} \tag{3.1a}$$

$$B_{ij}^{st} = b_i^s b_j^t \tag{3.1b}$$

$$C_{ij}^{st} = \tfrac{1}{2}(b_i^{\dagger s} b_j^t + b_j^t b_i^{\dagger s}) \tag{3.1c}$$

with

$$[b_i^{\dagger s}, b_j^{\dagger t}] = [b_i^s, b_j^t] = 0 \qquad [b_i^s, b_j^{\dagger t}] = \delta_{ij}\delta_{st}$$

for $i, j = 1, 2, \ldots, n$ and $s, t = 1, 2, \ldots, 2\sigma$.

More precisely, the harmonic oscillator representation is a unitary infinite-dimensional representation of the double covering group (called the metaplectic group) of Sp($2\sigma n$, R) which is reducible into two fundamental irreps denoted by

$$\langle 1/2(0)\rangle \qquad \text{and} \qquad \langle 1/2(1)\rangle. \tag{3.2}$$

The notation used in (3.2) expresses the fact that these irreps can be built from lowest-weight states of the representations of the maximal compact subgroup U($2\sigma n$) with lowest weights

$$\langle 1/2(0)\rangle \sim (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \ldots) \qquad \langle 1/2(1)\rangle \sim (\tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \ldots)$$

respectively.

Under the restriction

$$\mathrm{Sp}(2\sigma n, R) \downarrow \mathrm{Sp}(n, R) \times \mathrm{O}(2\sigma)$$

the two subgroups, Sp(n, R) and O(2σ) are complementary (Moshinsky and Quesne 1971, Kashiwara and Vergne 1978). This implies that the irreps of O(2σ) are uniquely determined by the irreps of Sp(n, R) and vice-versa. Then, one has the branching rule

$$\begin{aligned} &\mathrm{Sp}(2\sigma n, R) \downarrow \mathrm{Sp}(n, R) \times \mathrm{O}(2\sigma) \\ &\langle 1/2(0)\rangle + \langle 1/2(1)\rangle \downarrow \sum_{\lambda} \langle \sigma(\lambda)\rangle \times [\lambda] \end{aligned} \tag{3.3}$$

where $\langle \sigma(\lambda)\rangle$ is a character of an Sp(n, R) irrep, $[\lambda]$ the character of the corresponding O(2σ) irrep and the sum includes partitions (λ) for which $(\tilde{\lambda}) = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots)$ satisfies the constraints $\tilde{\lambda}_1 + \tilde{\lambda}_2 \leqslant 2\sigma$ and $\tilde{\lambda}_1 \leqslant n$.

Following Rowe *et al* (1985), the U(n) content of an Sp(n, R) irrep $\langle \sigma(\lambda)\rangle$ is inferred by comparing the two branching rules

$$\mathrm{Sp}(2\sigma n, R) \downarrow \mathrm{Sp}(n, R) \times \mathrm{O}(2\sigma) \downarrow \mathrm{U}(n) \times \mathrm{O}(2\sigma) \tag{3.4a}$$

$$\mathrm{Sp}(2\sigma n, R) \downarrow \mathrm{U}(2\sigma n) \downarrow \mathrm{U}(n) \times \mathrm{O}(2\sigma) \tag{3.4b}$$

resulting in the branching rule

$$\begin{aligned} &\mathrm{Sp}(n, R) \downarrow \mathrm{U}(n) \\ &\langle \sigma(\lambda)\rangle \downarrow \varepsilon^{\sigma}\{\lambda_s . D\}_k \end{aligned} \tag{3.5}$$

where

(i) ε^σ is the $\mathrm{U}(1) \subset \mathrm{U}(n)$ character; $\varepsilon^\sigma(g) = (\det g)^\sigma$;

(ii) $\{\lambda_s\}$ is a signed sequence of S-functions

$$\{\lambda_s\} = \{\lambda\} + \sum \pm\{\rho\}$$

with the sum including those S-functions $\{\rho\}$ for which the corresponding $\mathrm{O}(2\sigma)$ irreps $\pm[\rho]$ are equivalent to $[\lambda]$ under the modification rules of $\mathrm{O}(2\sigma)$. For example, for $2\sigma = A$ and $n > A$

$$\{2_s\} = \{2\} - \{22^{A-1}\}$$

(iii) the symbol in (3.5) means product of the S-functions of the signed sequence by those of the D series;

(iv) the index $k = \min(2\sigma, n)$ indicates that only S-functions with up to k parts are to be retained in the product.

Note that a signed sequence $\{\lambda_s\}$ reduces to its leading term $\{\lambda\}$ when $\sigma > n$. However, in order to account for those cases where non-trivial signed sequences occur, it is more convenient for present purposes to label an $\mathrm{Sp}(n, R)$ irrep by $\langle\sigma\{\lambda_s\}\rangle$ instead of using, as in (3.3), the notation $\langle\sigma(\lambda)\rangle$ of Rowe *et al* (1985). So for example, what in the former notation would be the $\mathrm{Sp}(3, R)$ irrep $\langle 1(2)\rangle$ one labels now by $\langle 1(\{2\} - \{22\})\rangle$. By this simple device, the $\mathrm{Sp}(n, R) \supset \mathrm{U}(n)$ characters become much more tractable.

In the new notation, the branching rule (3.5a) becomes

$$\begin{aligned}&\mathrm{Sp}(n, R) \downarrow \mathrm{U}(n)\\ &\langle\sigma\{\lambda_s\}\rangle \downarrow \varepsilon^\sigma\{\lambda_s . D\}_k\end{aligned} \tag{3.6a}$$

and its inverse

$$\begin{aligned}&\mathrm{U}(n) \uparrow \mathrm{Sp}(n, R)\\ &\varepsilon^\sigma\{\lambda_s . D\}_k \downarrow \langle\sigma\{\lambda_s . D.C\}_k\rangle = \langle\sigma\{\lambda_s\}\rangle\end{aligned} \tag{3.6b}$$

where in (3.6b) use was made of the fact that D and C are inverse series.

It is known that the $\langle\sigma\{\lambda_s\}\rangle$ $\mathrm{Sp}(n, R)$ irreps arising in the branching rule (3.3) are in one-to-one correspondence with those obtained in the reduction of the $\mathrm{Sp}(n, R)$ tensor representation

$$\underbrace{(\langle 1/2\{0\}\rangle + \langle 1/2\{1\}\rangle)(\langle 1/2\{0\}\rangle + \langle 1/2\{1\}\rangle) \ldots (\langle 1/2\{0\}\rangle + \langle 1/2\{1\}\rangle)}_{2\sigma \text{ times}}. \tag{3.7}$$

However, what is needed in practical applications is a knowledge of which irreps occur in suitably symmetrised products of representations corresponding to systems of particles that are identical to within, for example, spin and/or isospin differences. What we need is to construct many-particle spatial wavefunctions of well-defined permutation symmetry (as well as $\mathrm{Sp}(n, R)$ symmetry) that can subsequently be combined with spin isospin wavefunctions, of contragredient permutation symmetry, to form totally antisymmetric states in accordance with the requirement of the Pauli principle. To achieve this objective, we make use of the fact that the symmetric group, $\mathrm{S}_{2\sigma}$, of permutations of $A = 2\sigma$ particle space coordinates is a subgroup of $\mathrm{O}(2\sigma)$. Thus we consider the branching rule

$$\begin{aligned}&\mathrm{Sp}(2\sigma n, R) \downarrow \mathrm{Sp}(n, R) \times \mathrm{O}(2\sigma) \downarrow \mathrm{Sp}(n, R) \times \mathrm{S}_{2\sigma}\\ &\langle 1/2(0)\rangle + \langle 1/2(1)\rangle \downarrow \sum_\lambda \langle\sigma\{\lambda_s\}\rangle \times [\lambda] \downarrow \sum_{\lambda f} c_{\lambda f}\langle\sigma\{\lambda_s\}\rangle \times \{f\}.\end{aligned} \tag{3.8}$$

The task to be done is then to determine the $c_{\lambda f}$ coefficients giving the $\langle\sigma\{\lambda_s\}\rangle$ irreps that occur with a given permutation symmetry $\{f\}$. To my knowledge this task has not been previously attempted.

The desired coefficients can be obtained in one of two ways.

(i) Direct method: evaluation of the plethysm

$$(\langle 1/2\{0\}\rangle+\langle 1/2\{1\}\rangle)\otimes\{f\}. \tag{3.9}$$

Then if $\{f\}$ denotes the desired space permutation symmetry of the $A=2\sigma$ particle states, the $\mathrm{Sp}(n, R)$ irreps obtained in the reduction of the above plethysm are in one-to-one correspondence with the irreps $\langle\sigma\{\lambda_s\}\rangle$ of (3.8).

(ii) Indirect method: given an $\mathrm{Sp}(n, R)$ irrep, $\langle\sigma\{\lambda_s\}\rangle$, one can determine with which permutation symmetries it occurs, and their multiplicities (coefficients $c_{\lambda f}$), by exploiting the complementarity between $\mathrm{Sp}(n, R)$ and $\mathrm{O}(2\sigma)$ and making use of the branching rule $\mathrm{O}(2\sigma)\downarrow \mathrm{S}_{2\sigma}$ (Dehuai and Wybourne 1981).

The first method is described in detail in the next section. An example of application of the second method is given in subsection 5.2.

4. The plethysm $(\langle 1/2\{0\}\rangle+\langle 1/2\{1\}\rangle)\otimes\{f\}$ for $\mathrm{Sp}(n, R)$

The method for evaluating normal or symmetrised Kronecker products of $\mathrm{Sp}(n, R)$ representations will be, first, to express their characters in terms of S-functions by branching down to $\mathrm{U}(n)$, then to perform the corresponding operations on the S-functions and finally to invert the branching rule in order to return a result in terms of $\mathrm{Sp}(n, R)$ characters.

The S-function expansion of the fundamental irrep of $\mathrm{Sp}(n, R)$ is simply the M series, i.e.

$$\begin{aligned}&\mathrm{Sp}(n, R)\downarrow \mathrm{U}(n)\\ &\langle 1/2\{0\}\rangle+\langle 1/2\{1\}\rangle\downarrow \varepsilon^{1/2}(\{0\}.D+\{1\}.D)_1=\varepsilon^{1/2}M\end{aligned} \tag{4.1}$$

where $(\{0\}.D)_1=M_+$, $(\{1\}.D)_1=M_-$ and $M_++M_-=M$ (cf (2.15)).

Then for the plethysm one has

$$\begin{aligned}&\mathrm{Sp}(n, R)\downarrow \mathrm{U}(n)\uparrow \mathrm{Sp}(n, R)\\ &(\langle 1/2\{0\}\rangle+\langle 1/2\{1\}\rangle)\otimes\{f\}\downarrow \varepsilon^{\sigma}(M\otimes\{f\})\uparrow\langle\sigma\{M\otimes\{f\}.C\}_k\rangle\end{aligned}$$

with $k=\min(2\sigma, n)$. It follows then that

$$\sum_{\lambda} c_{\lambda f}\{\lambda_s\}=\{M\otimes\{f\}.C\}_k \tag{4.2}$$

and all one has to do is to obtain the S-function content of $M\otimes\{f\}.C$. The appropriate route here is to follow the method of McConnell and Newell (1973).

According to McConnell and Newell, the S-function content of a series can be found by converting its polynomial generating function into a sum of determinantal ratios. For example, the C series given by (cf (2.14a)),

$$C=\prod_{i\leq j}^{k}(1-\alpha_i\alpha_j)=\det(\alpha_t^{k-s}-\alpha_t^{k+s})/\det(\alpha_t^{k-s}) \tag{4.3}$$

corresponds to the sum

$$\sum\pm\begin{Bmatrix}0 & 0 & 0 & 0 & \dots & 0\\ 2 & 4 & 6 & 8 & \dots & 2k\end{Bmatrix} \tag{4.4a}$$

of 2^k Schur functions, not all necessarily standard, obtained by putting 0 or $2l$ for the lth entry, $1 \leqslant l \leqslant k$, and the negative sign being affixed only when an odd number of entries $2, 4, 6, \ldots, 2k$ are taken. For example, the notation $\{{}^{0}_{2}\}$ means $\{{}^{0}_{2}\} = \{0\} - \{2\}$. Using modification rules on the non-standard S-functions of (4.4a) again yields the familiar expansion

$$C = \sum_{\gamma} (-1)^{|\gamma|/2} \{\gamma\} \tag{4.4b}$$

limited to partitions with up to k parts.

For instance, for $k = 3$ (4.4a) gives

$$C = \{000\} - \{200\} - \{040\} + \{240\} - \{006\} + \{206\} + \{046\} - \{246\}.$$

Reducing the parts in each S-function to descending order, one obtains, as expected.

$$C = \{0\} - \{2\} + \{31\} - \{33\} - \{411\} + \{431\} - \{442\} + \{444\}.$$

Now, the S-function content of any other generating polynomial function, of which C is a factor, can be determined by simply multiplying the rows of the determinant in the numerator of (4.3) by the corresponding terms of the co-factor polynomial.

By way of example consider

$$\prod_{i}^{k} (1 - \alpha_i)^{-1} \prod_{i \leqslant j}^{k} (1 - \alpha_i \alpha_j) \tag{4.5}$$

which is the generating function of the product series $M.C.$ Its S-function content is obtained in the following manner

$$\begin{aligned} M.C &= \prod_{i}^{k} \left(\sum_{m} \alpha_i^m \right) \prod_{i \leqslant j}^{k} (1 - \alpha_i \alpha_j) \\ &= \prod_{t}^{k} \left(\sum_{m} \alpha_t^m \right) \det(\alpha_t^{k-s} - \alpha_t^{k+s})[\det(\alpha_t^{k-s})]^{-1} \\ &= \det\left[\sum_{m} (\alpha_t^{m+k-s} - \alpha_t^{m+k+s}) \right] [\det(\alpha_t^{k-s})]^{-1}. \end{aligned} \tag{4.6}$$

Expanding the determinant in the numerator with respect to s and m,

$$M.C = \det(\alpha_t^{k-1} + \alpha_t^k, \alpha_t^{k-2} + \alpha_t^{k-1} + \alpha_t^k + \alpha_t^{k+1}, \ldots, \alpha_t^0 + \alpha_t^1 + \ldots + \alpha_t^{2k-1}) / \det(\alpha_t^{k-s})$$

and rearranging the columns of the determinant by subtracting linear combinations of earlier columns

$$M.C = \det(\alpha_t^{k-1} + \alpha_t^k, \alpha_t^{k-2} + \alpha_t^{k+1}, \ldots, \alpha_t^0 + \alpha_t^{2k-1}) / \det(\alpha_t^{k-s}). \tag{4.7}$$

Finally, using (i) the property that a determinant where the elements of a column are the sums of a like number of terms is equal to the sum of the determinants in which the elements of the column in question are replaced by the individual terms, and (ii) the definition (2.1) extended to non-standard S-functions, one obtains from (4.7) the finite series

$$M.C = \sum \left\{ \begin{matrix} 0 & 0 & 0 & \ldots & 0 \\ 1 & 3 & 5 & \ldots & 2k-1 \end{matrix} \right\} \tag{4.8}$$

where the sum includes 2^k S-functions for which the lth entry is either 0 or $2l - 1$ with $1 \leqslant l \leqslant k$.

It is straightforward to check that after application of the modification rules to (4.8) the result is the series G restricted to S-functions with up to k parts, as expected. Indeed the result in this case could have been obtained much more quickly by simply reducing the product of the generating functions of M and C to the generating function of G (Yang and Wybourne 1986); i.e.

$$\begin{aligned} M.C &= \prod_{i}^{k} (1-\alpha_i)^{-1} \prod_{i\leqslant j}^{k} (1-\alpha_i\alpha_j) \\ &= \prod_{i}^{k} (1-\alpha_i)^{-1} \prod_{i}^{k} (1-\alpha_i^2) \prod_{i<j}^{k} (1-\alpha_i\alpha_j) \\ &= \prod_{i}^{k} (1+\alpha_i) \prod_{i<j}^{k} (1-\alpha_i\alpha_j) \\ &= G. \end{aligned} \tag{4.9}$$

For the trivial case $\{f=1\}$, we have $2\sigma=1$ and hence $k=1$. The series G restricted to S-functions having only one part is the series

$$G_1 = \{0\} + \{1\}.$$

Therefore one obtains the obvious result

$$(\langle 1/2\{0\}\rangle + \langle 1/2\{1\}\rangle) \otimes \{1\} = \langle 1/2\{G_1\}\rangle. \tag{4.10}$$

The evaluation of the plethysm $\{M\otimes\{f\}.C\}_k$ with $\{f\}$ different from $\{1\}$ involves the following considerations.

(i) $\{f\}$ is an irrep of the symmetric group $S_{2\sigma}$, so one uses (2.5) to expand it in terms of the power sum functions.

(ii) Plethysm is an operation distributive on the right with respect to addition and multiplication. So one has

$$\begin{aligned} M\otimes\{f\}.C &= (1/(2\sigma)!)(\textstyle\sum h_\rho \chi_\rho^{(f)} M\otimes S_\rho).C \\ &= (1/(2\sigma)!)(\textstyle\sum h_\rho \chi_\rho^{(f)} (M\otimes p_1)^{\nu_1} (M\otimes p_2)^{\nu_2} \ldots (M\otimes p_{2_\sigma})^{\nu_{2\sigma}}).C. \end{aligned} \tag{4.11}$$

For example, let $2\sigma=4$. Then $\{f\}$ can be either $\{4\}$, $\{31\}$, $\{22\}$, $\{211\}$ or $\{1^4\}$. For each possible $\{f\}$ it follows from (4.11) and the character table of S_4 that

$$\begin{aligned} &(M\otimes\{4\}).C \\ &\quad = \tfrac{1}{24}[(M\otimes p_1)^4.C + 6(M\otimes p_1)^2(M\otimes p_2).C + 8(M\otimes p_1)(M\otimes p_3).C \\ &\qquad + 3(M\otimes p_2)(M\otimes p_2).C + 6(M\otimes p_4).C] \end{aligned}$$

$$\begin{aligned} &(M\otimes\{31\}).C \\ &\quad = \tfrac{1}{24}[3(M\otimes p_1)^4.C + 6(M\otimes p_1)^2(M\otimes p_2).C \\ &\qquad - 3(M\otimes p_2)(M\otimes p_2).C - 6(M\otimes p_4).C] \end{aligned}$$

$$\begin{aligned} &(M\otimes\{22\}).C \\ &\quad = \tfrac{1}{24}[2(M\otimes p_1)^4.C - 8(M\otimes p_1)(M\otimes p_3).C + 6(M\otimes p_2)(M\otimes p_2).C] \end{aligned}$$

$$\begin{aligned} &(M\otimes\{211\}).C \\ &\quad = \tfrac{1}{24}[3(M\otimes p_1)^4.C - 6(M\otimes p_1)^2(M\otimes p_2).C \\ &\qquad - 3(M\otimes p_2)(M\otimes p_2).C + 6(M\otimes p_4).C] \end{aligned}$$

$$(M\otimes\{1^4\}).C$$
$$=\tfrac{1}{24}[(M\otimes p_1)^4.C-6(M\otimes p_1)^2(M\otimes p_2).C+8(M\otimes p_1)(M\otimes p_3).C$$
$$+3(M\otimes p_2)(M\otimes p_2).C-6(M\otimes p_4).C]. \tag{4.12}$$

The explicit expansion of these plethysms is given, for Sp(3, R), in subsection 5.2.

In general, one has to evaluate two typical terms.

Term 1.

$$(M\otimes p_m)^n.C. \tag{4.13}$$

Making use of the property that

$$P(\alpha)\otimes p_r=P(\alpha^r)$$

where $P(\alpha)$ is a polynomial in the α_i, we have

$$\{(M\otimes p_m)^n.C\}_k$$
$$=\prod_i^k(1-\alpha_i^m)^{-n}\prod_{i\leq j}^k(1-\alpha_i\alpha_j)$$
$$=\left(\prod_t^k\sum_q\alpha_t^{qm}\right)^n\det(\alpha_t^{k-s}-\alpha_t^{k+s})[\det(\alpha_t^{k-s})]^{-1}$$
$$=\prod_t^k\left(\sum_{q_1}\alpha_t^{q_1m}\right)\left(\sum_{q_2}\alpha_2^{q_2m}\right)\ldots\left(\sum_{q_n}\alpha_t^{q_nm}\right)\det(\alpha_t^{k-s}-\alpha_t^{k+s})[\det(\alpha_t^{k-s})]^{-1}$$
$$=\det\left(\sum_{q_i^{(s)}}(\alpha_t^{q_1^{(s)}m+\ldots+q_n^{(s)}m+k-s}-\alpha_t^{q_1^{(s)}m+\ldots+q_n^{(s)}m+k+s}\right)[\det(\alpha_t^{k-s})]^{-1} \tag{4.14}$$

which following the method illustrated before, will yield the series of S-functions

$$\sum\pm\left\{\begin{matrix}\sum_i^n q_i^{(1)}m & \sum_i^n q_i^{(2)}m & \ldots & \sum_i^n q_i^{(k)}m\\ \sum_i^n q_i^{(1)}m+2 & \sum_i^n q_i^{(2)}m+4 & \ldots & \sum_i^n q_i^{(k)}m+2k\end{matrix}\right\} \tag{4.15}$$

obtained by putting either $\Sigma_i^n q_i^{(l)}m$ or $\Sigma_i^n q_i^{(l)}m+2l$ for the lth entry and affixing the minus sign when an odd number of $\Sigma_i q_i^{(l)}m+2l$ entries are taken.

Note that the parameters $q_i^{(s)}$ can take integer values from zero to infinity independently and thus the same S-function may appear more than once. For example, the S-function $\{m, 0, \ldots 0\}$ appears with multiplicity n in the above expansion.

Term 2.

$$(M\otimes p_{m_1})\ldots(M\otimes p_{m_n}).C \tag{4.16}$$

$$\{(M\otimes p_{m_1})\ldots(M\otimes p_{m_n}).C\}_k$$
$$=\prod_i^k(1-\alpha_i^{m_1})^{-1}\ldots\prod_i^k(1-\alpha_i^{m_n})^{-1}\prod_{i\leq j}^k(1-\alpha_i\alpha_j)$$
$$=\prod_t^k\sum_{q_1}(\alpha_t^{q_1m_1})\ldots\prod_t^k\sum_{q_n}(\alpha_t^{q_nm_n})\det(\alpha_t^{k-s}-\alpha_t^{k+s})[\det(\alpha_t^{k-s})]^{-1}$$
$$=\det\left(\sum_{q_i^{(s)}}(\alpha_t^{q_1^{(s)}m_1+\ldots+q_n^{(s)}m_n+k-s}-\alpha_t^{q_1^{(s)}m_1+\ldots+q_n^{(s)}+k+s})\right)[\det(\alpha_t^{k-s})]^{-1} \tag{4.17}$$

which corresponds to the following sum of S-functions:

$$\sum \pm \begin{Bmatrix} \sum_i^n q_i^{(1)} m_i & \sum_i^n q_i^{(2)} m_i & \ldots & \sum_i^n q_i^{(k)} m_i \\ \sum_i^n q_i^{(1)} m_i + 2 & \sum_i^n q_i^{(2)} m_i + 4 & \ldots & \sum_i^n q_i^{(k)} m_i + 2k \end{Bmatrix}. \tag{4.18}$$

Again the S-functions are such that the lth entry is either $\Sigma_i^n q_i^{(l)} m_i$ or $\Sigma_i^n q_i^{(l)} m_i + 2l$ with the minus sign being affixed when an odd number of entries of the second type is taken. Here again, with the exception of the single S-function $\{0\}$, all the others may appear with multiplicity.

5. Illustrative examples for Sp(3, R)

Due to the importance, in physical applications, of the symplectic group in three dimensions, explicit examples of the expansion of $\langle \sigma\{M \otimes \{f\}.C\}_k\rangle$ are given, in terms of Sp(3, R) characters.

5.1. Two-particle case in Sp(3, R)

The tensor product $(\langle 1/2\{0\}\rangle + \langle 1/2\{1\}\rangle)(\langle 1/2\{0\}\rangle + \langle 1/2\{1\}\rangle)$ of Sp(3, R) irreps contains symmetric $\{2\}$ and antisymmetric $\{1^2\}$ parts. If the product corresponds to a coupled system of identical particles it should be either one or the other. It follows then that in order to know which Sp(3, R) irreps are compatible with each permutation symmetry one has to evaluate the two following symmetrised products:

$$\{M \otimes \{2\}.C\}_2 = \tfrac{1}{2}(M \otimes p_1^2.C + M \otimes p_2.C) \tag{5.1}$$

$$\{M \otimes \{1^2\}.C\}_2 = \tfrac{1}{2}(M \otimes p_1^2.C - M \otimes p_2.C) \tag{5.2}$$

where here the maximum allowed number of parts, k, for the S-functions is only two (recall $k = \min(2, 3)$).

Particularising (4.15) for $m = 1$ and $n = 2$, one has that

$$\{M \otimes p_1^2.C\}_2 = \{q_1 + q_2, q_1' + q_2'\} \tag{5.3a}$$

$$-\{q_1 + q_2, q_1' + q_2' + 4\} \tag{5.3b}$$

$$-\{q_1 + q_2 + 2, q_1' + q_2'\} \tag{5.3c}$$

$$+\{q_1 + q_2 + 2, q_1' + q_2' + 4\} \tag{5.3d}$$

where q_1, q_2, q_1' and q_2' can take any integer value independently.

It is straightforward to verify that the following hold true.

(a) $\{0\}$ can only occur once ($q_1 = q_2 = q_1' = q_2' = 0$) from (5.3$a$); its multiplicity is 1.

(b) $\{1\}$ occurs twice ($q_1 = 1$; $q_2 = q_1' = q_2' = 0$ or $q_2 = 1$; $q_1 = q_1' = q_2' = 0$) from (5.3a); its multiplicity is 2.

(c) Any S-function $\{n\}$ with $n \geqslant 2$ occurs only from terms (5.3a) and (5.3c); its multiplicity equals the difference between the number of times it occurs in (5.3a) and in (5.3c), which is always 2.

(d) $\{11\}$ occurs from (5.3a) with multiplicity 4; however, its multiplicity is reduced to 1 since the non-standard S-function $\{02\} = -\{11\}$ occurs with multiplicity 3, also from the same term (5.3a).

(e) The multiplicity of an S-function of the type $\{n, 1\}$, where $n > 1$, is zero. This is so because the standard $\{n, 1\}$ S-function occurs as many times as the non-standard $\{0, n+1\}$, resulting in total cancellation.

(f) All S-functions of the type $\{m, 2\}$ with $n \geqslant 2$ appear with multiplicity -2. Consider, for example, $\{32\}$

from (5.3a) $\quad 12\{32\}+10\{14\}=12\{32\}-10\{32\}=2\{32\}$

from (5.3b) $\quad -2\{14\}=2\{32\}$

from (5.3c) $\quad -6\{32\}$.

Total $= -2\{32\}$.

(g) Contributions from (5.3) to any S-function $\{n, m\}$, with $n \geqslant m$ and $m \geqslant 3$, add up to zero.

Thus one has

$$\begin{aligned}\{(M\otimes p_1)^2.C\}_2 &= \{0\}+2\{1\}+\{11\}+2\{2\}-2\{22\}+2\{3\}-2\{32\}\\ &\quad +2\{4\}-2\{42\}+2\{5\}-2\{52\}+\dots\end{aligned} \tag{5.4a}$$

Now, particularising (4.18) for $m=2$ and $n=1$, one obtains

$$\{(M\otimes p_2.C)\}_2=\{2q, 2q'\}-\{2q, 2q'+4\}-\{2q+2, 2q'\}+\{2q+2, 2q'+4\}$$

and the only S-functions that survive are $\{0\}$ and $\{11\}$. Thus

$$\{(M\otimes p_2.C)\}_2=\{0\}-\{11\}. \tag{5.4b}$$

Substituting (5.4a) and (5.4b) into (5.1) and (5.2), one obtains the Sp(3, R) results (cf (4.2)):

$$\begin{aligned}(\langle 1/2\{0\}\rangle+\langle 1/2\{1\}\rangle)\otimes\{2\} &= \langle 1\{0\}\rangle+\langle 1\{1\}\rangle+\langle 1(\{2\}-\{22\})\rangle+\langle 1(\{3\}-\{32\})\rangle\\ &\quad +\langle 1(\{4\}-\{42\})\rangle+\langle 1(\{5\}-\{52\})\rangle+\dots\end{aligned}$$

$$\begin{aligned}(\langle 1/2\{0\}\rangle+\langle 1/2\{1\}\rangle)\otimes\{11\} &= \langle 1\{11\}\rangle+\langle 1\{1\}\rangle+\langle 1\{2\}-\{22\}\rangle\rangle+\langle 1(\{3\}-\{32\})\rangle\\ &\quad +\langle 1(\{4\}-\{42\})\rangle+\langle 1(\{5\}-\{52\})\rangle+\dots\end{aligned}$$

The two-particle case is simple enough for an analytical method to be used as an alternative to derive the above results. Note that $M\otimes p_1 = M$, so that

$$(M\otimes p_1)^2.C = MMC.$$

Expressing MMC in terms of the indeterminates α_i, one has

$$MMC=\prod_i^2(1-\alpha_i)^{-2}\prod_{i\leqslant j}^2(1-\alpha_i\alpha_j)=\prod_i^2(1-\alpha_i)^{-1}(1+\alpha_i)(1-\alpha_1\alpha_2).$$

But $\Pi_i(1-\alpha_i)^{-1}(1+\alpha_i)$ is the generating function (Yang and Wybourne 1986) of the S-function series

$$S_2=\{0\}+2\sum_{n\geqslant 1}\{n\}+2\sum_{n\geqslant 1}\{n, 1\}$$

i.e. the series S restricted to partitions with not more than two parts and

$$(1-\alpha_1\alpha_2)=\{0\}-\{1^2\}.$$

Thus

$$MMC = S_2.(\{0\}-\{1^2\}) = \{0\}+\{1^2\}+2\{1\}+2\sum_{n\geq 2}(\{n\}-\{n,2\}). \qquad (5.5)$$

On the other hand, $M\otimes p_2.C = MM^{+}C$ (recall (2.15a)) is simply

$$MM^{+}C=\prod_{i}^{2}(1-\alpha_i^2)^{-1}\prod_{i\leq j}^{2}(1-\alpha_i\alpha_j)=(1-\alpha_1\alpha_2)=\{0\}-\{1^2\}. \qquad (5.6)$$

So the expansions (5.4) have been obtained again.

Summarising then, one has that the Sp(3, R) irreps for a system of two particles which occur with permutational particle symmetry $\{2\}$ are

$$\langle 1\{0\}\rangle, \langle 1\{1\}\rangle, \langle 1(\{2\}-\{22\}\rangle, \langle 1(\{3\}-\{32\})\rangle, \langle 1(\{4\}-\{42\})\rangle, \ldots, \langle 1(\{n\}-\{n2\})\rangle, \ldots$$

and with permutational particle symmetry $\{1^2\}$

$$\langle 1\{1\}\rangle, \langle 1\{1^2\}\rangle, \langle 1(\{2\}-\{22\}\rangle, \langle 1(\{3\}-\{32\})\rangle, \langle 1(\{4\}-\{42\})\rangle, \ldots, \langle 1(\{n\}-\{n2\})\rangle, \ldots.$$

These irreps are listed in increasing order of their excitation energy in the harmonic oscillator shell model space. The excitation energy $n\hbar\omega$ of an irrep $\langle 1\{\lambda_s\}\rangle$ is defined as the energy of its lowest-weight state and is given by $n=\lambda_1+\lambda_2+\lambda_3$ of the leading S-function $\{\lambda\}$ of the signed sequence $\{\lambda_s\}$.

The lowest-weight state of an Sp(3, R) irrep is defined as the state that satisfies the conditions

$$B_{ij}\Phi_{\mathrm{LWS}}=0 \qquad i,j=1,2,3$$

$$C_{ij}\Phi_{\mathrm{LWS}}=0 \qquad j>i=1,2,3$$

where $B_{ij}=\Sigma_s\, b_i^s b_j^s$ and $C_{ij}=\frac{1}{2}\Sigma_s\,(b_i^{+s}b_j^s+b_j^s b_i^{+s}), j>i$, are the lowering operators of Sp(3, R).

A two-particle lowest-weight state for an Sp(3, R) irrep $\langle 1\{n_s\}\rangle$ of excitation energy $n\hbar\omega$ can be written

$$\Phi_{\mathrm{LWS}}=\sum_{k=0}^{(n-1)/2} a_k\varphi_{2k+1}^{(1)}\varphi_{n-2k-1}^{(2)} \qquad (5.7)$$

where $\varphi_k^{(s)}$ are normalised single-particle harmonic oscillator wavefunctions in one dimension. To ensure that (5.7) is indeed a lowest-weight state, one has only to require that

$$B_{11}\Phi_{\mathrm{LWS}}=0.$$

Since

$$b_1^s b_1^s\varphi_k^{(t)}=\sqrt{k!/(k-2)!}\;\varphi_{k-2}^{(s)}\delta_{st}$$

one has then

$$B_{11}\Phi_{\mathrm{LWS}}=\sum_{k}^{(n-1)/2} a_k[\sqrt{(2k+1)!/(2k-1)!}\;\varphi_{2k-1}^{(1)}\varphi_{n-2k-1}^{(2)} + \sqrt{(n-2k-1)!/(n-2k-3)!}\;\varphi_{2k+1}^{(1)}\varphi_{n-2k-3}^{(2)}]$$

which will vanish for a particular combination of the coefficients a_k.

Application of permutation operators to such a lowest-weight state will yield an appropriately symmetrised state.

5.2. *Four-particle system*

The ^{4}He nucleus is already of interest to the nuclear symplectic shell model. In this section, the lowest excited collective spaces, of definite particle symmetry, of the ^{4}He configuration space will be identified. To achieve this objective, one needs the following partial results.

(i)

$$\begin{aligned}\{(M\otimes p_1)^4.C\}_3 &= \{n_1+n_2+n_3+n_4,\ n_1'+n_2'+n_3'+n_4',\ n_1''+n_2''+n_3''+n_4''\}\\ &\quad -\{n_1+n_2+n_3+n_4,\ n_1'+n_2'+n_3'+n_4',\ n_1''+n_2''+n_3''+n_4''+6\}\\ &\quad -\{n_1+n_2+n_3+n_4,\ n_1'+n_2'+n_3'+n_4'+4,\ n_1''+n_2''+n_3''+n_4''\}\\ &\quad +\{n_1+n_2+n_3+n_4,\ n_1'+n_2'+n_3'+n_4'+4,\ n_1''+n_2''+n_3''+n_4''+6\}\\ &\quad -\{n_1+n_2+n_3+n_4+2,\ n_1'+n_2'+n_3'+n_4',\ n_1''+n_2''+n_3''+n_4''\}\\ &\quad +\{n_1+n_2+n_3+n_4+2,\ n_1'+n_2'+n_3'+n_4',\ n_1''+n_2''+n_3''+n_4''+6\}\\ &\quad +\{n_1+n_2+n_3+n_4+2,\ n_1'+n_2'+n_3'+n_4'+4,\ n_1''+n_2''+n_3''+n_4''\}\\ &\quad -\{n_1+n_2+n_3+n_4+2,\ n_1'+n_2'+n_3'+n_4'+4,\ n_1''+n_2''+n_3''+n_4''+6\}\end{aligned}$$

where n_i, n_i', n_i'' $(i=1,\ldots,4)$ can take any integer value from zero to infinity independently.

Up to weight 4, the S-function expansion is explicitly

$$\begin{aligned}\{(M\otimes p_1)^4.C\}_3 &= \{0\}+4\{1\}+6\{11\}+9\{2\}+16\{3\}+16\{21\}+4\{111\}+25\{4\}+30\{31\}\\ &\quad +10(\{22\}-\{222\})+9\{211\}+\ldots.\end{aligned} \tag{5.8}$$

(ii)

$$\begin{aligned}\{(M\otimes p_3)(M\otimes p_1).C\}_3 &= \{3n_1+n_2,\ 3n_1'+n_2',\ 3n_1''+n_2''\}\\ &\quad -\{3n_1+n_2,\ 3n_1'+n_2',\ 3n_1''+n_2''+6\}\\ &\quad -\{3n_1+n_2,\ 3n_1'+n_2'+4,\ 3n_1''+n_2''\}\\ &\quad +\{3n_1+n_2,\ 3n_1'+n_2'+4,\ 3n_1''+n_2''+6\}\\ &\quad -\{3n_1+n_2+2,\ 3n_1'+n_2',\ 3n_1''+n_2''\}\\ &\quad +\{3n_1+n_2+2,\ 3n_1'+n_2',\ 3n_1''+n_2''+6\}\\ &\quad +\{3n_1+n_2+2,\ 3n_1'+n_2'+4,\ 3n_1''+n_2''\}\\ &\quad -\{3n_1+n_2+2,\ 3n_1'+n_2'+4,\ 3n_1''+n_2''+6\}\end{aligned}$$

where n_i, n_i', n_i'' $(i=1,2)$ can take any integer value from zero to infinity independently.

Up to weight 4 the S-function expansion is explicitly

$$\begin{aligned}\{(M\otimes p_3)(M\otimes p_1).C\}_3 &= \{0\}+\{1\}+0\{11\}+0\{2\}+\{3\}-2\{21\}+\{111\}+\{4\}+0\{31\}\\ &\quad -2(\{22\}-\{222\})+0\{211\}+\ldots.\end{aligned} \tag{5.9}$$

(iii)

$$\{(M\otimes p_2)^2.C\}_3$$
$$\begin{aligned}
&=\{2n_1+2n_2,2n_1'+2n_2',2n_1''+2n_2''\}\\
&\quad-\{2n_1+2n_2,2n_1'+2n_2',2n_1''+2n_2''+6\}\\
&\quad-\{2n_1+2n_2,2n_1'+2n_2'+4,2n_1''+2n_2''\}\\
&\quad+\{2n_1+2n_2,2n_1'+2n_2'+4,2n_1''+2n_2''+6\}\\
&\quad-\{2n_1+2n_2+2,2n_1'+2n_2',2n_1''+2n_2''\}\\
&\quad+\{2n_1+2n_2+2,2n_1'+2n_2',2n_1''+2n_2''+6\}\\
&\quad+\{2n_1+2n_2+2,2n_1'+2n_2'+4,2n_1''+2n_2''\}\\
&\quad-\{2n_1+2n_2+2,2n_1'+2n_2'+4,2n_1''+2n_2''+6\}
\end{aligned}$$

where n_i, n_i', n_i'' $(i=1,2)$ can take any integer value from zero to infinity independently.

Up to weight 4, the S-function expansion is explicitly

$$\{(M\otimes p_2)^2.C\}_3$$
$$\begin{aligned}
&=\{0\}+0\{1\}+\{2\}-2\{11\}+0\{3\}+0\{21\}+0\{111\}+\{4\}-2\{31\}\\
&\quad+2(\{22\}-\{222\})+\{211\}+\dots
\end{aligned}\tag{5.10}$$

(iv)

$$\{(M\otimes p_2)(M\otimes p_1)^2.C\}_3$$
$$\begin{aligned}
&=\{2n_1+n_2+n_3,2n_1'+n_2'+n_3',2n_1''+n_2''+n_3''\}\\
&\quad-\{2n_1+n_2+n_3,2n_1'+n_2'+n_3',2n_1''+n_2''+n_3''+6\}\\
&\quad-\{2n_1+n_2+n_3,2n_1'+n_2'+n_3'+4,2n_1''+n_2''+n_3''\}\\
&\quad+\{2n_1+n_2+n_3,2n_1'+n_2'+n_3'+4,2n_1''+n_2''+n_3''+6\}\\
&\quad-\{2n_1+n_2+n_3+2,2n_1'+n_2'+n_3',2n_1''+n_2''+n_3''\}\\
&\quad+\{2n_1+n_2+n_3+2,2n_1'+n_2'+n_3',2n_1''+n_2''+n_3''+6\}\\
&\quad+\{2n_1+n_2+n_3+2,2n_1'+n_2'+n_3'+4,2n_1''+n_2''+n_3''\}\\
&\quad-\{2n_1+n_2+n_3+2,2n_1'+n_2'+n_3'+4,2n_1''+n_2''+n_3''+6\}
\end{aligned}$$

where n_i, n_i', n_i'' $(i=1,2,3)$ can take any integer value from zero to infinity independently.

Up to weight 4 the S-function expansion is explicitly

$$\{(M\otimes p_2)(M\otimes p_1)^2.C\}_3$$
$$\begin{aligned}
&=\{0\}+2\{1\}+3\{2\}+0\{11\}+4\{3\}+0\{21\}-2\{111\}+5\{4\}+0\{31\}\\
&\quad+0(\{22\}+\{222\})-3\{211\}+\dots
\end{aligned}\tag{5.11}$$

(v)

$$\{(M\otimes p_4).C\}_3$$
$$\begin{aligned}
&=\{4n,4n',4n''\}-\{4n,4n',4n''+6\}-\{4n,4n'+4,4n''\}\\
&\quad+\{4n,4n'+4,4n''+6\}-\{4n+2,4n',4n''\}+\{4n+2,4n',4n''+6\}\\
&\quad+\{4n+2,4n'+4,4n''\}-\{4n+2,4n'+4,4n''+6\}
\end{aligned}$$

where n, n', n'' can take any integer value from zero to infinity independently.

Up to weight 4, the S-function expansion is explicitly

$$\{(M\otimes p_4).C\}_3$$
$$=\{0\}+0\{1\}-\{2\}+0\{11\}+0\{3\}+0\{21\}+0\{111\}+\{4\}$$
$$+0\{31\}+0(\{22\}-\{222\})+\{211\}+\dots \tag{5.12}$$

The coefficient associated with a particular S-function, in each one of the expansions (5.8)-(5.12), can be easily computed. By way of example, the method followed in the calculation of the coefficient of the S-function $\{222\}$, is given in table form. Table 1 gives the number of combinations of n_1, n_2, n_3, n_4 for which their sum, N, is $N=0,\dots,4$. Contributions to the total coefficient of $\{222\}$ also come from the non-standard S-functions $\{213\}$, $\{132\}$, $\{033\}$, $\{114\}$ and $\{024\}$. Their occurrence in each term (i)-(v) is given in table 2.

Combining now results of equations (5.8)-(5.12), according to equations (4.12), one finally obtains

$$(\langle 1/2\{0\}\rangle+\langle 1/2\{1\}\rangle)\otimes\{4\}$$
$$=\langle 2\{0\}\rangle+\langle 2\{1\}\rangle+\langle 2\{2\}\rangle+2\langle 2\{3\}\rangle+3\langle 2\{4\}\rangle+\langle 2\{31\}\rangle+\dots$$

$$(\langle 1/2\{0\}\rangle+\langle 1/2\{1\}\rangle)\otimes\{31\}$$
$$=\langle 2\{1\}\rangle+2\langle 2\{2\}\rangle+\langle 2\{11\}\rangle+3\langle 2\{3\}\rangle+2\langle 2\{21\}\rangle+4\langle 2\{4\}\rangle$$
$$+4\langle 2\{31\}\rangle+\langle 2(\{22\}-\{222\})\rangle+\dots$$

$$(\langle 1/2\{0\}\rangle+\langle 1/2\{1\}\rangle)\otimes\{22\}$$
$$=\langle\{2\}\rangle+\langle 2\{3\}\rangle+2\langle 2\{21\}\rangle+2\langle 2\{4\}\rangle+2\langle 2\{31\}\rangle$$
$$+2\langle 2(\{22\}+\{222\})\rangle-\langle 2\{211\}\rangle+\dots$$

Table 1. Number of combinations of n_1, n_2, n_3, n_4 for which the sums in each column add up to $N=0,\dots,4$.

N	$n_1+n_2+n_3+n_4$	$3n_1+n_2$	$2n_1+2n_2$	$2n_1+n_2+n_3$	$4n$
0	1	1	1	1	1
1	4	1	0	2	0
2	10	1	2	4	0
3	20	2	0	6	0
4	35	2	3	9	1

Table 2. Occurrence, in each term (i)-(v), of the S-functions that lead to the multiplicity of $\{222\}$ in each expansion (5.8)-(5.12).

	(i)	(ii)	(iii)	(iv)	(v)
$\{222\}$	900	0	4	48	0
$\{132\}=-\{222\}$	800	2	0	48	0
$\{213\}=-\{222\}$	720	0	0	36	0
$\{033\}=+\{222\}$	400	4	0	36	0
$\{114\}=+\{222\}$	560	2	0	36	0
$\{024\}=-\{222\}$	350	2	6	36	0
Total	-10	$+2$	-2	0	0

$$(\langle 1/2\{0\}\rangle+\langle 1/2\{1\}\rangle)\otimes\{21^2\}$$
$$=\langle 2\{11\}\rangle+\langle 2\{3\}\rangle+2\langle 2\{21\}\rangle+\langle 2\{111\}\rangle+2\langle 2\{4\}\rangle+4\langle 2\{31\}\rangle$$
$$+\langle 2(\{22\}-\{222\})\rangle+2\langle 2\{211\}\rangle+\ldots$$
$$(\langle 1/2\{0\}\rangle+\langle 1/2\{1\}\rangle)\otimes\{1^4\}=\langle 2\{31\}\rangle+\langle 2\{211\}\rangle+\ldots$$

where all Sp(3, R) representations of ^{4}He, up to harmonic oscillator excitation $4\hbar\omega$, are displayed.

Thus one concludes, for instance, that the ground-state collective space of ^{4}He, labelled by $\langle 2\{0\}\rangle$, can only occur with permutational symmetry $\{4\}$. Likewise, the collective space of excitation $4\hbar\omega$ labelled by $\langle 2(\{22\}-\{222\})\rangle$ can occur with symmetries $\{31\}$ and $\{211\}$, once, and with symmetry $\{22\}$ twice.

The above results can be checked, term by term, using the indirect method mentioned in section 3. This method exploits the complementarity between Sp(n, R) and O(2σ); in this case, Sp(3, R) and O(4). Thus, for each Sp(3, R) irrep $\langle 2\{\lambda_s\}\rangle$ there is an O(4) irrep $[\lambda]$, and if $[\lambda]$ is compatible with certain permutational symmetries, so is its Sp(3, R) counterpart. So, if, for example, one wants to know with which symmetries one can construct the Sp(3,R) irrep $\langle 2(\{22\}-\{222\})\rangle$, one just has to make use of the branching rule $\mathrm{O}(4)\downarrow \mathrm{S}_4$ for [22].

The branching rules for $\mathrm{O}(n)\downarrow \mathrm{S}_n$ have been studied by Dehuai and Wybourne (1981), who gave an n-independent prescription for their evaluation. According to their results, the O(4) character [22] is expressed in S_4 characters by

$$[22]\downarrow\langle 22\rangle+2\langle 21\rangle+\langle 3\rangle+3\langle 2\rangle+\langle 11\rangle+\langle 1\rangle$$

where the characters of the symmetric group are given in reduced notation. Converting them to characters of S_4 and using modification rules on the non-standard ones

$$\langle 22\rangle=\{022\}=-\{122\}=0$$
$$\langle 21\rangle=\{121\}=0$$
$$\langle 3\rangle=\{13\}=-\{22\}$$
$$\langle 2\rangle=\{22\}$$
$$\langle 11\rangle=\{211\}$$
$$\langle 1\rangle=\{31\}$$

one gets the expected result

$$\mathrm{O}(4)\downarrow \mathrm{S}_4 \qquad [22]\downarrow\{31\}+2\{22\}+\{211\}.$$

In conclusion, the technique required for the evaluation of symmetrised Kronecker products of the fundamental representation of Sp(n, R) has been given, and general formulae produced. The solution of this problem is of particular relevance to the identification of the collective spaces of a given nucleus in the framework of the symplectic shell model. Further work on this problem will include the automation of the calculation of the coefficients $c_{\lambda f}$.

Acknowledgments

The author wishes to express her sincere thanks to Professors B G Wybourne, R King and D J Rowe for very stimulating discussions. She is also indebted to the 'Summer

Institute in Theoretical Physics' for extending an invitation to her, and for the financial support.

References

Black G R E, King R C and Wybourne B G 1983 *J. Phys. A: Math. Gen.* **16** 1555
Carvalho J, LeBlanc R, Vassanji M and Rowe D J 1986 *Nucl. Phys.* A **452** 263
Dehuai L and Wybourne B G 1981 *J. Phys. A: Math. Gen.* **14** 327
Kashiwara M and Vergne M 1978 *Inventions Math.* **44** 1
King R C 1975 *J. Phys. A: Math. Gen.* **8** 429
King R C, Luan Dehuai and Wybourne B G 1981 *J. Phys. A: Math. Gen.* **14** 2509
King R C and Wybourne B G 1982 *J. Phys. A: Math. Gen.* **15** 1137
Littlewood D E 1940 *The Theory of Group Characters* (Oxford: Clarendon) (2nd edition 1950)
McConnell J and Newell M J 1973 *Proc. R. Irish Acad.* A **73** 255
Moshinsky M and Quesne C 1971 *J. Math. Phys.* **12** 1772
Murnaghan F D 1938 *Theory of Group Representations* (Baltimore: Johns Hopkins)
Rowe D J 1985 *Rep. Prog. Phys.* **48** 1419
Rowe D J, Wybourne B G and Butler P H 1985 *J. Phys. A: Math. Gen.* **18** 939
Weyl H 1939 *Classical Groups* (Princeton, NJ: Princeton University Press)
Wybourne B G 1970 *Symmetry Principles and Atomic Spectroscopy* (New York: Wiley)
Yang M and Wybourne B G 1986 *J. Phys. A: Math. Gen.* **19** 3513